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Realizations of Multimode Quantum Group $SU(2)_{q,s}$

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By virtue of the concept of the two-parameter deformed multimode bosonic oscillator, the Nodvik and Holstein–Primakoff realizations of the two-parameter deformed multimode quantum group $SU(2)_{q,s}$ are given. The deformed mappings between the multimode quantum group $SU(2)_{q,s}$ and the two-parameter deformed multimode bosonic oscillator are also presented.

In the past few years, quantum groups and algebras have been shown to arise in many problems of physical and mathematical interest. Much effort is now being devoted to the construction of their representations, and recently many realizations have been usefully devised using the q-deformation and q,s-deformation of single-mode bosonic operators (Faddeev, 1981; Drinfeld, 1986; Jimbo, 1986; Kulish *et al.*, 1981; Biedenharn, 1989; Macfarlane, 1989; Sun *et al.*, 1989; Yan, 1990; Ng, 1990; Katriel *et al.*, 1991; Nodvik, 1969; Demidov *et al.*, 1990; Sudbery, 1990; Schirrmacher *et al.*, 1991; Burdik *et al.*, 1991; Chakrabarti *et al.*, 1991; Jing, 1993; Zhou *et al.*, 1995; Curtright *et al.*, 1990; Song, 1990; Quesne, 1991; Hu, 1992; Mallick *et al.*, 1991; Yu *et al.*, n.d.). In this paper we introduce the concept of the q, s-deformed multimode bosonic oscillator, and derive the Nodvik and Holstein–Primakoff realizations of the multimode quantum group $SU(2)_{q,s}$ and give the deformed mappings between the multimode quantum group $SU(2)_{q,s}$ and the q, s-deformed multimode bosonic oscillators.

We introduce independent two groups of q, s-deformed bosonic oscillators

$$\{a_i^+, a_i, n_i^a\} \quad \text{and} \quad \{b_i^+, b_i, n_i^b\} \qquad \text{for} \quad i = 1, 2, \ldots, k$$

They satisfy the commutation relations (Jing, 1993)

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$$a_i^+ a_i = [n_i^a]_{q,s}, \qquad a_i a_i^+ = [n_i^a + 1]_{q,s},$$

$$[n_i^a, a_i^+] = a_i^+, \qquad [n_i^a, a_i] = -a_i \tag{1a}$$

$$a_i a_i^+ - s^{-1} q a_i^+ a_i = (sq)^{-n_i^a}, \qquad a_i a_i^+ - (sq)^{-1} a_i a_i^+ = (s^{-1}q)^{n_i^a} \tag{1b}$$

$$b_i^+ b_i = [n_i^b]_{q,s^{-1}}, \qquad b_i b_i^+ = [n_i^b + 1]_{q,s^{-1}},$$

$$[n_i^b, b_i^+] = b_i^+, \qquad [n_i^b, b_i] = -b_i \tag{1c}$$

$$b_i b_i^+ - sq b_i^+ b_i = (sq^{-1})^{n_i^b} \tag{1d}$$

where we have used the notations $[x]_{q,s} = s^{1-x}(q^x - q^{-x})/q - q^{-1})$ and $[x]_{q,s^{-1}} = s^{x-1}(q^x - q^{-x})/q - q^{-1})$.
The x can be operators or general numbers.

We define two independent q, s-deformed k-mode bosonic operators as follows:

$$A_k = a_1 a_2 \cdots a_k \left\{ \frac{[n_1^a]_{q,s} [n_2^a]_{q,s} \cdots [n_k^a]_{q,s}}{\min([n_1^a]_{q,s}, [n_2^a]_{q,s}, \ldots, [n_k^a]_{q,s})} \right\}^{-1/2} \tag{2a}$$

$$B_k = b_1 b_2 \cdots b_k \left\{ \frac{[n_1^b]_{q,s^{-1}} [n_2^b]_{q,s^{-1}} \cdots [n_k^b]_{q,s^{-1}}}{\min([n_1^b]_{q,s^{-1}}, [n_2^b]_{q,s^{-1}}, \ldots, [n_k^b]_{q,s^{-1}})} \right\}^{-1/2} \tag{2b}$$

It is easy to check that

$$A_k A_k^+ - s^{-1} q A_k^+ A_k = (sq)^{-N_k^a}, \qquad A_k A_k^+ - (sq)^{-1} A_k^+ A_k = (s^{-1}q)^{N_k^a} \tag{3a}$$

$$[N_k^a, A_k^+] = A_k^+, \qquad [N_k^a, A_k] = -A_k \tag{3b}$$

$$B_k B_k^+ - sq\, B_k^+ B_k = (sq^{-1})^{N_k^b} \tag{3c}$$

$$[N_k^b, B_k^+] = B_k^+, \qquad [N_k^b, B_k] = -B_k \tag{3d}$$

where the symbols N_k^a and N_k^b are given by

$$N_k^a = \min(n_1^a, n_2^a, \ldots, n_k^a) \tag{4a}$$

$$N_k^b = \min(n_1^b, n_2^b, \ldots, n_k^b) \tag{4b}$$

It is easy to find that $\{A_k^+, A_k, N_k^a\}$ and $\{B_k^+, B_k, N_k^b\}$ denote a q, s-deformed k-mode bosonic oscillator.

The generators of the two-parameter deformed k-mode quantum group $SU(2)_{q,s}$ can be obtained from the Jordan–Schwinger realization in terms of the q, s-deformed k-mode bosonic creation and annihilation operators (Yu *et al.*, n.d.)

$$J_k^+ = A_k^+ B_k, \qquad J_k^- = B_k^+ A_k, \qquad J_k^0 = \tfrac{1}{2}(N_k^a - N_k^b) \tag{5}$$

They satisfy the following commutation relations:

$$[J_k^0, J_k^{\pm}] = \pm J_k^{\pm}, \qquad s^{-1}J_k^+J_k^- - sJ_k^-J_k^+ = s^{-2J_k^0}[2J_k^0] \tag{6}$$

The k-mode quantum group $SU(2)_{q,s}$ is a Hopf algebra; its coproduct, antipode, and counit are, respectively, as follows.

Coproduct:

$$\Delta(J_k^0) = J_k^0 \otimes 1 + 1 \otimes J_k^0 \tag{7a}$$

$$\Delta(J_k^{\pm}) = (sq)^{-J_k^0} \otimes J_k^{\pm} + J_k^{\pm} \otimes (sq^{-1})^{-J_k^0} \tag{7b}$$

$$\Delta(1) = 1 \otimes 1 \tag{7c}$$

Antipode:

$$S(J_k^0) = -J_k^0 \tag{8a}$$

$$S(J_k^+) = -(sq)J_k^+ \, s^{2J_k^0} \tag{8b}$$

$$S(J_k^-) = -(sq^{-1})J_k^- \, s^{2J_k^0} \tag{8c}$$

Counit:

$$\mathscr{E}(J_k^0) = \mathscr{E}(J_k^{\pm}) = 0 \tag{9a}$$

$$\mathscr{E}(1) = 1 \tag{9b}$$

The Hilbert space basis determined by equation (3a)–(3d) is

$$|n, n, \ldots\rangle = \frac{1}{\sqrt{[n]_{q,s}!}}(A_k^+)^n|0, 0, \ldots\rangle \tag{10a}$$

$$|\widetilde{m, m}, \ldots\rangle = \frac{1}{\sqrt{[m]_{q,s^{-1}}!}}(B_k^+)^m|0, 0, \ldots\rangle \tag{10b}$$

where $|n, n, \ldots\rangle = |n\rangle_1|n\rangle_2 \cdots |n\rangle_k$ and $|\widetilde{m, m}, \ldots\rangle = |\tilde{m}\rangle_1|\tilde{m}_2\rangle \cdots |\tilde{m}\rangle_k$. The actions of the operators $\{A_k^+, A_k, N_k^a\}$ and $\{B_k^+, B_k, N_k^b\}$ on equations (10) are, respectively,

$$A_k^+|n, n, \ldots\rangle = \sqrt{[n+1]_{q,s}}\,|n+1, n+1, \ldots\rangle \tag{11a}$$

$$A_k|n, n, \ldots\rangle = \sqrt{[n]_{q,s}}\,|n-1, n-1, \ldots,$$

$$N_k^a|n, n, \ldots\rangle = n|n, n, \ldots\rangle \tag{11b}$$

and

$$B_k^+|\widetilde{m, m}, \ldots\rangle = \sqrt{[m+1]_{q,s^{-1}}}|\widetilde{m+1}, m+1, \ldots\rangle \tag{12a}$$

$$B_k|m, m, \ldots\rangle = \sqrt{[m]_{q,s^{-1}}}|\widetilde{m-1}, m-1, \ldots\rangle, \tag{12b}$$

$$N_k^b|\widetilde{m, m}, \ldots\rangle = m|\widetilde{m, m}, \ldots\rangle$$

The unitary irreducible representation basis $|j, m; j, m; \ldots\rangle$ of the k-mode quantum group $SU(2)_{q,s}$ is

$$|j,m;j,m;\ldots\rangle = |j+m, j+m,\ldots\rangle \otimes |j-m, j-m,\ldots\rangle = \frac{(A_k^+)^{j+m}}{\sqrt{[j+m]_{q,s}!}}|0,0,\ldots\rangle \otimes \frac{(B_k^+)^{j-m}}{\sqrt{[j-m]_{q,s^{-1}}!}}|0,0,\ldots\rangle \quad (-j \le m \le j) \tag{13}$$

where $|j, m; j, m; \ldots\rangle = |j, m\rangle_1 |j, m\rangle_2 \ldots |j, m\rangle_k$. These irreducible representations are finite and depend on a single quantum number $j = 0, 1/2, 1, \ldots$. The actions of the k-mode quantum group $SU(2)_{q,s}$ generators on the elements of the irreducible representation (13) are given by

$$J_k^+ |j, m; j, m; \ldots\rangle = \sqrt{[j-m]_{q,s^{-1}} [j+m+1]_{q,s}} \times |j, m+1; j, m+1; \ldots\rangle \tag{14a}$$

$$J_k^- |j, m; j, m; \ldots\rangle = \sqrt{[j+m]_{q,s} [j-m+1]_{q,s^{-1}}} \times |j, m-1; j, m-1; \ldots\rangle \tag{14b}$$

$$J_k^0 |j, m; j, m; \ldots\rangle = m\, |j, m; j, m; \ldots\rangle \tag{14c}$$

The Casimir operator of the k-mode quantum group $SU(2)_{q,s}$ is

$$C = s^{2J_k^0}(J_k^+ J_k^- + s^{-2}[J_k^0]_{q,s}[J_k^0 - 1]_{q,s}) = s^{2J_k^0}(s^2 J_k^- J_k^+ + [J_k^0]_{q,s}[J_k^0 + 1]_{q,s}) = s^{2J_k^0}[J_k]_{q,s}[J_k + 1]_{q,s} \tag{15}$$

$$C|j, m; j, m; \ldots\rangle = s^{2j}[j]_{q,s}[j+1]_{q,s}|j, m; j, m; \ldots\rangle \tag{16}$$

According to the above properties of the k-mode quantum group $SU(2)_{q,s}$, it is easy to find its Nodvik realization in the form

$$J_k^+ = e^{-iP_k}\, s^{(j-u_k-1)}\sqrt{[j-u_k]_{q,s}[j+u_k+1]_{q,s}} \tag{17a}$$

$$J_k^- = \sqrt{[j-u_k]_{q,s}\,[j+u_k+1]_{q,s}}\, s^{(j-u_k-1)} e^{iP_k} \tag{17b}$$

$$J_k^0 = u_k$$

where u_k and P_k is a canonical commutator pair, i.e., $[u_k, P_k] = i$. It is easy to check that (17a)–(17c) satisfy (6).

For the q, s-deformed k-mode bosonic oscillator one has the following realizations:

$$A_k = e^{-iP_k}\sqrt{[j-u_k]_{q,s}}, \qquad A_k^+ = \sqrt{[j-u_k]_{q,s}}\, e^{iP_k} \tag{18a}$$

$$A_k^+ A_k = [j-u_k]_{q,s} \tag{18b}$$

and

$$B_k = e^{-iP_k}\sqrt{[j-u_k]_{q,s^{-1}}}, \qquad B_k^+ = \sqrt{[j-u_k]_{q,s^{-1}}}\, e^{iP_k} \tag{19a}$$

$$B_k^+ B_k = [j-u_k]_{q,s^{-1}} \tag{19b}$$

It is easy to find that (18) and (19) satisfy (3a) and (3c), respectively. From the above results, we can obtain the deformed mappings between the k-mode quantum group $SU(2)_{q,s}$ and the k-mode q, s-deformed bosonic oscillators as follows:

$$J_k^+ = \tilde{J}_k^+ f(J_k^0), \qquad J_k^- = f(J_k^0)\tilde{J}_k^-, \qquad J_k^0 = \tilde{J}_k^0 \tag{20}$$

and

$$A_k = \tilde{A}_k\sqrt{\frac{[N_k^a]_{q,s}}{N_k^a}}, \qquad A_k^+ = \sqrt{\frac{[N_k^a]_{q,s}}{N_k^a}}\,\tilde{A}_k^+ \tag{21a}$$

$$B_k = \tilde{B}_k\sqrt{\frac{[N_k^b]_{q,s^{-1}}}{N_k^b}}, \qquad B_k^+ = \sqrt{\frac{[N_k^b]_{q,s^{-1}}}{N_k^b}}\,\tilde{B}_k^+ \tag{21b}$$

with

$$f(J_k^0) = s^{(J_k^0-u_k-1)}\sqrt{\frac{[j-J_k^0]_{q,s}[j+J_k^0+1]_{q,s}}{(j-J_k^0)(j+J_k^0+1)}} \tag{22}$$

where the operators with the tildes in (20) and (21) denote the nondeformed ones. From (17)–(19), we can obtain the Holstein–Primakoff realization of the k-mode quantum group $SU(2)_{q,s}$

$$J_k^+ = s^{N_k^a}\sqrt{[2j-N_k^a]_{q,s}}A_k, \qquad J_k^- = A_k^+\sqrt{[2j-N_k^a]_{q,s}}s^{N_k^a} \tag{23a}$$

$$J_k^0 = j - N_k^a \tag{23b}$$

and

$$J_k^+ = s^{N_k^b}\sqrt{[2j-N_k^b]_{q,s^{-1}}}B_k, \qquad J_k^- = B_k^+\sqrt{[2j-N_k^b]_{q,s^{-1}}}s^{N_k^b} \tag{24a}$$

$$J_k^0 = j - N_k^b \tag{24b}$$

In this paper, we have considered some realizations of the k-mode quantum group $SU(2)_{q,s}$ in terms of the q,s-deformed k-mode bosonic oscillators. We believe that the method used above will be suitable for studying other multimode quantum groups.

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